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RIGHT-ANGLED COXETER QUANDLES AND POLYHEDRAL PRODUCTS (Algebraic Topology focused on Transformation Groups)

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RIGHT-ANGLED COXETER QUANDLES AND POLYHEDRAL PRODUCTS

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1. INTRODUCTION

This report is a survey of the paper [K].

A quandle is a set with a binary operation satisfying three conditions which axiomatize conjugation. In particular, there are mutually adjoint constructions of quandles and groups through conjugation: (1) we associate a quandle to a conjugation closed subset of a group; (2) we associate a group to a quandle. Thus by composing these two constructions, we can construct a new group out of a given conjugation closed subset of a group. However, the interest of the resulting group entirely depends on the choice of a conjugation closed subset of a group. One of good choices is the set of reflection of a Coxeter system (W, S) [N], and we call the resulting group the adjoint group of W . The adjoint group of W is given by an explicit presentation, but its structure has not been known except for the case when W is the symmetric group [AFGV, E]. Recently, Akita [A] investigates the natural connection between the adjoint group of W and the underlying Coxeter group W to describe the structures of the adjoint group of W . We will study the classifying space of the adjoint group of W based on Akita's result [A] in terms of spaces called polyhedral products when W is right-angled.

2. COXETER QUANDLES AND AKITA'S RESULT

2.1. Coxeter systems. A pair (W, S) of a group W and a finite set S is called a *Coxeter system* if it is equipped with a map $m: S \times S \rightarrow \{1, 2, \dots, \infty\}$ satisfying the following three conditions:

- (1) $m(s, t) = m(t, s)$;
- (2) $m(s, t) = 1$ if and only if $s = t$;
- (3) the group W has a presentation

$$W = \langle s \in S \mid (st)^{m(s,t)} = 1 \text{ for } m(s, t) < \infty \rangle.$$

Hereafter, (W, S) will denote a Coxeter system. Our reference of Coxeter groups is [D].

The Artin group associated with a Coxeter system (W, S) is defined by

$$A_W = \langle a_s (s \in S) \mid \underbrace{a_s a_t a_s \cdots}_{m(s,t)} = \underbrace{a_t a_s a_t \cdots}_{m(t,s)} \text{ for } 1 < m(s, t) < \infty \rangle.$$

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Then the Coxeter group W is obtained by adding the relations $a_s^2 = 1$ to A_W and replacing a_s with s for $s \in S$ so that there is a natural projection

$$\pi: A_W \rightarrow W, \quad a_s \mapsto s.$$

2.2. Coxeter quandles. Recall that a *quandle* X is a set equipped with a binary operation $*$: $X \times X \rightarrow X$ satisfying the following three conditions:

- (1) $x * x = x$;
- (2) $(x * y) * z = (x * z) * (y * z)$;
- (3) the map $X \rightarrow X, x \mapsto x * w$ is bijective for any $w \in X$.

We consider the following two mutually adjoint constructions producing groups from quandles and vice versa. To a quandle X there is associated a group

$$(2.1) \quad \text{Ad}(X) = \langle e_x \ (x \in X) \mid e_{x*y} = e_y^{-1} e_x e_y \rangle$$

which is called the *adjoint group*, and a conjugation closed subset R of a group G is regarded as a quandle by the binary operation

$$*: R \times R \rightarrow R, \quad g * h = h^{-1}gh.$$

Then in particular, from a given conjugation closed subset X of a group one gets a new group $\text{Ad}(X)$. and moreover, it is easy to verify that the map

$$\phi: \text{Ad}(X) \rightarrow G, \quad e_x \mapsto x$$

is a well-defined surjection.

An element of the form $w^{-1}sw$ for $w \in W, s \in S$ is called a reflection of W , and we denote by X_W the set of all reflections of W . Then X_W is a quandle by the above construction and we call it the *Coxeter quandle* associated with a Coxeter system (W, S) . Thus we get a group $\text{Ad}(X_W)$, the classifying space of which is our object to study.

2.3. Akita's results. In [A], Akita proved several properties of $\text{Ad}(X_W)$, and these results are summarized as follows.

Theorem 2.1. *The commutative square*

$$\begin{array}{ccc} \text{Ad}(X_W) & \xrightarrow{\text{ab}} & \mathbb{Z}^{c(W)} \\ \downarrow \phi & & \downarrow \text{proj} \\ W & \xrightarrow{\text{ab}} & (\mathbb{Z}/2)^{c(W)} \end{array}$$

is a pullback, where $c(W)$ is the integer explicitly defined by W .

Since there is a commutative diagram

$$\begin{array}{ccc} A_W & \xrightarrow{\text{ab}} & \mathbb{Z}^{c(W)} \\ \downarrow \pi & & \downarrow \text{proj} \\ W & \xrightarrow{\text{ab}} & (\mathbb{Z}/2)^{c(W)}, \end{array}$$

we have:

Corollary 2.2 (Akita [A]). *The map $\pi: A_W \rightarrow W$ factors as the composite of surjections*

$$A_W \rightarrow \text{Ad}(X_W) \xrightarrow{\phi} W.$$

Theorem 2.1 passes to a homotopy pullback by taking the classifying spaces.

Theorem 2.3. *There is a homotopy pullback*

$$\begin{array}{ccc} B\text{Ad}(X_W) & \longrightarrow & (S^1)^{c(W)} \\ \downarrow \phi & & \downarrow \text{incl} \\ BW & \longrightarrow & (\mathbb{R}P^\infty)^{c(W)}. \end{array}$$

3. RIGHT-ANGLED COXETER GROUPS AND POLYHEDRAL PRODUCTS

3.1. Right-angled Coxeter groups. A Coxeter system (W, S) is called *right-angled* if $m(s, t) = 1, 2, \infty$ for any $s, t \in S$. We characterize the right-angularity of a Coxeter system (W, S) in terms of graph products of groups. Let G be a group and Γ be a simple graph with the vertex set V . The graph product of G_Γ is the quotient of the free product of $G_v = G$ for $v \in V$ by the commutator relations $[G_u, G_v] = 1$ whenever the vertices u, v are adjacent in Γ . Graph products of groups are natural in the sense that for a homomorphism $G \rightarrow H$ of groups and a subgraph inclusion $\Theta \rightarrow \Gamma$, there are homomorphisms

$$G_\Gamma \rightarrow H_\Gamma \quad \text{and} \quad G_\Theta \rightarrow G_\Gamma.$$

We regard a set S to be a discrete graph so that the graph product \mathbb{Z}_S is identified with the free group generated by S . We then set

$$\theta: \mathbb{Z}_S \rightarrow W, \quad \tilde{\theta}: \mathbb{Z}_S \rightarrow A_W, \quad \bar{\theta}: \mathbb{Z}_S \rightarrow \text{Ad}(X_W)$$

to be the projections induced from the inclusions of generators.

Define a graph Γ_W for a Coxeter system (W, S) such that its vertex set is S and $s, t \in S$ are adjacent in Γ_W whenever $m(s, t) = 2$. Then if (W, S) is right-angled, we have

$$W = \langle s \in S \mid s^2 = 1, [s, t] = 1 \text{ for } m(s, t) = 2 \rangle \cong (\mathbb{Z}/2)_{\Gamma_W}.$$

Moreover, taking the generating set S into account, we get:

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Proposition 3.1. *A Coxeter system (W, S) is right-angled if and only if there is an isomorphism*

$$W \cong (\mathbb{Z}/2)_{\Gamma_W}$$

such that the map $\theta: \mathbb{Z}_S \rightarrow W$ is identified with the projection $\mathbb{Z}_S \rightarrow (\mathbb{Z}/2)_{\Gamma_W}$.

If a Coxeter system (W, S) is right-angled, we also have

$$A_W = \langle s \in S \mid [s, t] = 1 \text{ for } m(s, t) = 2 \rangle \cong \mathbb{Z}_{\Gamma_W}.$$

Then we similarly get:

Proposition 3.2. *If a Coxeter system (W, S) is right-angled, there is an isomorphism*

$$A_W \cong \mathbb{Z}_{\Gamma_W}$$

such that the map $\tilde{\theta}: \mathbb{Z}_S \rightarrow A_W$ is identified with the projection $\mathbb{Z}_S \rightarrow \mathbb{Z}_{\Gamma_W}$.

3.2. Polyhedral products. We translate Proposition 3.1 into homotopy theory by using spaces called polyhedral products. Let (X, A) be a pair of spaces and K be an abstract simplicial complex on the vertex set $[m]$. We associate to $\sigma \in K$, possibly empty, a subspace $D(\sigma)$ of X^m such that

$$(3.1) \quad D_{(X,A)}(\sigma) = Y_1 \times \cdots \times Y_m, \quad Y_i = \begin{cases} X & i \in \sigma \\ A & i \notin \sigma. \end{cases}$$

The *polyhedral product* of (X, A) associated with K is now defined by

$$Z(K; (X, A)) = \bigcup_{\sigma \in K} D_{(X,A)}(\sigma).$$

Although polyhedral products are defined more generally for a sequence of pairs of spaces, a single pair of spaces is enough for our purpose here. We refer to [BBCG, IK] for the basic homotopy theory of polyhedral products.

We express the classifying space of a right-angled Coxeter group by a polyhedral product. A simplicial complex is called *flag* if every collection of pairwise adjacent vertices is its simplex. For a simple graph Γ , we denote by $\Delta(\Gamma)$ the flag complex whose 1-skeleton is Γ .

Proposition 3.3. *A Coxeter system (W, S) is right-angled if and only if there is a homotopy equivalence*

$$BW \simeq Z(\Delta(\Gamma_W); (\mathbb{R}P^\infty, *))$$

*such that the map $\theta: B\mathbb{Z}_S \rightarrow BW$ is identified with the inclusion $Z(S; (S^1, *)) \rightarrow Z(\Delta(\Gamma_W); (\mathbb{R}P^\infty, *))$.*

4. MAIN THEOREM AND ITS APPLICATIONS

4.1. Main theorem. By an old trick of homotopy theory, we can prove:

Lemma 4.1 (cf. [DS, Lemma 2.3.1]). *Let $(F, F') \rightarrow (E, E') \rightarrow B$ be a pair of homotopy fibrations where $(F, F'), (E, E')$ are NDR pairs. Then the sequence*

$$Z(K; (F, F')) \rightarrow Z(K; (E, E')) \rightarrow B^m$$

is a homotopy fibration.

Using this lemma, we can prove the following.

Proposition 4.2. *Let $F \rightarrow E \rightarrow B$ be a homotopy fibration such that the fiber inclusion is a cofibration. Then the commutative square*

$$(4.1) \quad \begin{array}{ccc} Z(K; (E, F)) & \xrightarrow{\text{incl}} & E^m \\ \downarrow & & \downarrow \\ Z(K; (B, *)) & \xrightarrow{\text{incl}} & B^m \end{array}$$

is a homotopy pullback, where m is the number of vertices of K .

We now prove the main theorem which indicates a homotopical inheritance of the right-angularity of a Coxeter system (W, S) by the adjoint group $\text{Ad}(X_W)$ as in Proposition 3.3. Let I_2 be the mapping cylinder of the degree 2 map $S^1 \rightarrow S^1$, and we consider the pair (I_2, S^1) such that the inclusion $S^1 \rightarrow I_2 \simeq S^1$ is of degree 2. Note that we also have an inclusion $S^1 \rightarrow I_2$ which is a homotopy equivalence.

Theorem 4.3. *If a Coxeter system (W, S) is right-angled, then there is a homotopy equivalence*

$$B\text{Ad}(X_W) \simeq Z(\Delta(\Gamma_W); (I_2, S^1))$$

*such that the map $\bar{\theta}: B\mathbb{Z}_S \rightarrow B\text{Ad}(X_W)$ is identified with the inclusion $Z(S; (S^1, *)) \rightarrow Z(\Delta(\Gamma_W); (I_2, S^1))$ induced from a homotopy equivalence $S^1 \xrightarrow{\sim} I_2$.*

4.2. Applications of Theorem 4.3. We first consider a torsion in the adjoint group $\text{Ad}(X_W)$. By definition, W always has a torsion whereas one of the biggest problems on Artin groups is to show whether they are torsion free or not [P]. Since $\text{Ad}(X_W)$ is an intermediate object between W and A_W by Theorem ??, one may ask:

Problem 4.4. *Is $\text{Ad}(X_W)$ torsion free?*

Suppose (W, S) is right-angled. Since $BA_W \simeq Z(\Delta(\Gamma_W); (S^1, *))$ is finite dimensional by Proposition 3.2, A_W is torsion free. So one may expect that $\text{Ad}(X_W)$ is also torsion free. Indeed, by Theorem 4.3 $B\text{Ad}(X_W) \simeq Z(\Delta(\Gamma_W); (I_2, S^1))$ is finite dimensional, so we get:

Corollary 4.5. *If a Coxeter system (W, S) is right-angled, then $\mathrm{Ad}(X_W)$ is torsion free.*

We next consider a property of the map $\Phi: A_W \rightarrow \mathrm{Ad}(X_W)$ when (W, S) is right-angled. We assume (W, S) is right-angled and $|S| = m$ so that we may identify S with $[m]$. Note that \mathbb{Z}^m is identified with the graph product $\mathbb{Z}_{\Delta^{m-1}}$. Then by Theorem 3.2, the map $\mathrm{ab}: BA_W \rightarrow B\mathbb{Z}^m$ is identified with the inclusion $Z(\Delta(\Gamma_W); (S^1, *)) \rightarrow Z(\Delta^{m-1}; (S^1, *)) = (S^1)^m$, so by Proposition ??, the map $\mathrm{ab}: BA_W \rightarrow B\mathbb{Z}^m$ is the projection

$$\Lambda_R(v_1, \dots, v_m) \rightarrow \Lambda_R(\Delta(\Gamma_W))$$

in cohomology. In particular, since $\Lambda_R(\Delta(\Gamma_W))$ is a free R -module, the map $\mathrm{ab}^*: H^*(\mathbb{Z}^m; R) \rightarrow H^*(A_W; R)$ has a section as R -modules. On the other hand, there is a commutative diagram

$$\begin{array}{ccc} A_W & \xrightarrow{\mathrm{ab}} & \mathbb{Z}^{c(W)} \\ \downarrow \Phi & & \parallel \\ \mathrm{Ad}(X_W) & \xrightarrow{\mathrm{ab}} & \mathbb{Z}^{c(W)} \end{array}$$

even when (W, S) is not right-angled. Thus by considering the induced commutative diagram in cohomology, we obtain:

Proposition 4.6. *If a Coxeter system (W, S) is right-angled, the map $\Phi^*: H^*(\mathrm{Ad}(X_W); R) \rightarrow H^*(A_W; R)$ has a section as R -modules.*

We will show that this property of Φ can be recovered by a homotopical property of the induced map $\Phi: BA_W \rightarrow B\mathrm{Ad}(X_W)$. To this end, we recall the natural homotopy decomposition of a suspension of a polyhedral product. Let (X, A) be a pair of spaces and K be a simplicial complex on the vertex set $[m]$. For any $\sigma \in K$, we put

$$\widehat{D}_{(X,A)}(\sigma) = Y_1 \wedge \cdots \wedge Y_m, \quad Y_i = \begin{cases} X & i \in \sigma \\ A & i \notin \sigma \end{cases}$$

and define the polyhedral smash product of (X, A) with respect to K by

$$\widehat{Z}(K; (X, A)) = \bigcup_{\sigma \in K} \widehat{D}_{(X,A)}(\sigma) \quad (\subset X^{\wedge n})$$

where $X^{\wedge n}$ is the smash product of n -copies of X . For a subset $I \subset [m]$, we put

$$K_I = \{\sigma \in K \mid \sigma \subset I\}.$$

Theorem 4.7 (Bahri, Bendersky, Cohen, and Gitler [BBCG]). *There is a homotopy equivalence*

$$\Sigma Z(K; (X, A)) \simeq \Sigma \bigvee_{\emptyset \neq I \subset [m]} \widehat{Z}(K_I; (X, A))$$

which is natural with respect to (X, A) .

Proposition 4.6 is recovered by the following homotopical property of the map $\Phi: A_W \rightarrow \text{Ad}(X_W)$.

Theorem 4.8. *If a Coxeter system (W, S) is right-angled, the map $\Sigma\Phi: \Sigma BA_W \rightarrow \Sigma B\text{Ad}(X_W)$ has a left homotopy inverse.*

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